

Intrinsic energy of Lemaître-Tolman-Bondi models and observational cosmology

R. Lapiedra^a and J. A. Morales-Lladosa^{a,1}

^aDepartament d'Astronomia i Astrofísica Universitat de València,
46100 Burjassot, València, Spain

E-mail: ramon.lapiedra@uv.es, antonio.morales@uv.es

Abstract. Recently, some Lemaître-Tolman-Bondi metrics have been considered as models alternative to the dark energy within the Friedmann-Lemaître-Robertson-Walker universes. One of these models has been confronted with cosmic observations, this confrontation seeming to rule it out. On the other hand, we show that the *intrinsic* energy of this model does not *physically* vanish. Thus we conclude that this model is actually a non *creatable* one. This suggests from the very beginning that the model is not a good model to represent our Universe, which would be in according with the negative result of this confrontation with observations.

¹Corresponding author.

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1 Introduction

Out of the particular case where $A' \neq 0$ (see next section), the Lemaître-Tolman-Bondi (LTB) metric family is the most general family of spherically symmetric metrics in General Relativity, corresponding to a pressure-less matter source [1].

Some of these metrics have been used, in a cosmological context, to describe large inhomogeneous structures and the anisotropies they produce on the cosmic background radiation temperature [2–4] and, more recently, as models alternative to the dark energy Friedmann-Lemaître-Robertson-Walker (FLRW) universes [5–11]. The models have been confronted with cosmological observations, and the final conclusion of one of these papers [11] is that the confrontation has become sufficiently constraining to "rule out the whole class of adiabatic LTB models", by testing a particular LTB model hereafter called the constrained García-Bellido-Haugbølle (CGBH) model.

On the other hand, from the beginning of the last seventies, people have speculated on the possibility that the Universe could have raised from a vacuum quantum fluctuation [12, 13], an idea further developed by Vilenkin [14]. If this had been the case, we could expect that the energy of our Universe, P^0 , and also the corresponding linear 3-momentum, P^i , and angular 4-momentum, $J^{\alpha\beta}$, vanish ($i, j, \dots = 1, 2, 3$ and $\alpha, \beta, \dots = 0, 1, 2, 3$). In [15], we have called such a universe with vanishing 4-momenta a *creatable* one. But it is well known that the energy and momenta of a space-time in General Relativity are dramatically dependent on the coordinates used. So, what coordinates must be used in order to calculate the specific two 4-momenta, P^α , $J^{\alpha\beta}$, that have to vanish in the case of a *creatable* universe? Our answer in [16] is that these specific coordinates have to be *intrinsic* ones, defined as follows:

1) First, they are Gauss coordinates in some finite region of the considered space-time, covering the 3-space boundary. That is, in this region, the metric components $g_{0\alpha}$ take the values $g_{00} = -1$, $g_{0i} = 0$. As it is well known, these are coordinates related to free falling observers.

2) Second, the corresponding linear 3-momentum, P^i , and angular 3-momentum, J^{ij} , vanish.

3) Third, let it be the space-like 3-surface $t = t_0 \equiv \text{const.}$, with t the time coordinate. We denote this 3-surface by Σ_3 . Then, asymptotically, the 3-space metric, g_{ij} , approaches fast enough a manifestly conformally flat metric ($g_{ij} = G^2 \delta_{ij}$) when we approach the boundary,

Σ_2 , of Σ_3 . Of course, if the space-time is asymptotically Minkowskian, the conformal factor is the unity.

Notice that in Gauss coordinates the time coordinate, t , is the proper time of those falling observers. Furthermore, is a *universal* time. This means that equal distant readings of this time correspond to events which are physically simultaneous (see [17], epigraph 84). This could be important in order to define a consistent energy, P^0 , since, as it is well known and we are going to see, its expression (see next (1.1)) is a 3-volume integral taken in a given instant t . That is, the elementary partial contributions to this integral are taken for the same t value, and it would be a good thing that this common instant labelled physically simultaneous events. Therefore, it would be good to deal with a proper and *universal* time, i.e., with Gauss coordinates, when calculating what we call here the *intrinsic* P^0 energy value: by definition, the one calculated in intrinsic coordinates. As far as the energy, P^0 , is concerned, the term *intrinsic* refers to the fact that this energy is, first, calculated for coordinates whose corresponding linear and angular 3-momenta, P^i and J^{ij} , vanish and, second, since these coordinates are associated to free falling observers (local inertial ones) that, in the present case of LTB metrics, are further co-moving with the source pressure-less matter. Thus, these observers add nothing extra to the considered space-time (compare this situation with, for example, the static observers in a static metric that would have to be prevented of free falling by some virtual non gravitational action). Therefore, we choose just this coordinates to define the specific P^0 energy and momenta, the *intrinsic* 4-momenta, that must vanish if a given metric is *creatable*. In the present case, if some LTB models are *creatable*.

These intrinsic coordinates can be proved to exist always [16] for each constant value $t = t_0$ and, as mentioned above, the corresponding linear and angular 4-momenta P^α and $J^{\alpha\beta}$ will be called *intrinsic* momenta. In all, we have called *creatable* universes the space-times whose *intrinsic* linear and angular 4-momenta, the last one irrespective of the momentum origin, vanish.

We will make explicit the existence of intrinsic coordinates in the particular case of the LTB and FLRW models considered in the present paper. As we will see next, the closed and flat FLRW universes have vanishing *intrinsic* momenta, while the value of the *intrinsic* energy of the open FLRW universe is $-\infty$. Thus, in our parlance, the two first ones are *creatable* universes, but the last one is not. Furthermore, the perturbed flat FRW universes in the frame of standard inflation in neither *creatable*, while the perturbed closed one remains *creatable* [18].

Let us precise that the starting point for our definition of the intrinsic momenta of a space-time is the Weinberg energy-momentum complex [19]. There are many complexes present in the literature on the subject, but in [20] we have explained why we have selected, among all them, the one from Weinberg, from which some specific expressions for the energy, P^0 , the linear 3-momentum, P^i , and angular 4-momentum, $J^{\alpha\beta}$, can be deduced. These expressions involve 3-space volume integrals that in the particular case of P^0 read

$$P^0 = \frac{1}{16\pi} \int \partial_i (\partial_j g_{ij} - \partial_i g) d^3x, \quad (1.1)$$

where the gravitational constant has been taken equal to 1, $g \equiv \delta_{ij} g_{ij}$ and the summation on repeated indices is performed with the Kronecker δ . This defined P^0 coincides (once writing it as a 2-surface integral on the above Σ_2 boundary) with the well known Arnowitt-Deser-Misner (ADM) energy [21].

Assuming that an existing universe has to be *creatable* is an attempt of saying something about our Universe all along its existence from just after the big bang. It can also supply us with a criteria for, from the beginning, discarding tentatively all non *creatable* metrics in General Relativity as good candidates for good cosmological models. This is just the case in the present paper where the CGBH model of the Universe is observationally discarded. This discarding could have been suggested from the very beginning because of the physically non *creatable* character of this model. We will prove this non *creatable* character along the present paper.

2 The LTB metrics and the general expression for their energy

The LTB metric class [1] can be written as [10]:

$$ds^2 = -dt^2 + \frac{A'(t,r)^2}{1-k(r)}dr^2 + A^2d\sigma^2 \equiv -dt^2 + dl^2, \quad (2.1)$$

with $d\sigma^2 = d\theta^2 + \sin^2\theta d\phi^2$ and where A and k are functions of the corresponding arguments satisfying the Einstein field equations, with $A' \equiv \partial_r A$ and $k < 1$. This family of metrics describes the spherical solutions of the Einstein field equations with a pressureless matter source [1]. We can see that the coordinates used are adapted to this spherical symmetry. Furthermore, they are Gauss coordinates, that is, for the metric components $g_{0\alpha}$ we have $g_{00} = -1$, $g_{0i} = 0$, such that the observers associated to these coordinates are free radially falling observers co-moving with the pressureless matter source.

We will assume that A' exists and is different from zero everywhere, except perhaps for $r = 0$. Note that from (2.1) we recover the FLRW metrics by putting $A = a(t)r$, $k = \kappa r^2$, with a the corresponding expansion factor and κ the curvature index, $\kappa = 1, 0, -1$.

Because of the manifest spherical symmetry of the metric (2.1), the linear 3-momentum, P^i , and the angular 3-momentum, J^{ij} , associated to this metric, vanish. Thus, the coordinates used in (2.1) will be intrinsic coordinates, as defined above, provided that the r coordinate be such that dl^2 becomes in the boundary of the corresponding 3-space a manifest conformally flat metric. Then, we are left with the question whether the energy, P^0 , calculated in such coordinates, that is, the *intrinsic* energy, vanishes or not in order to conclude if a particular LTB metric is *creatable* or not.

According to some general expressions given in [22], the expression for the P^0 energy of the metric (2.1) becomes

$$P^0 = \frac{1}{2} \lim_{r \rightarrow \infty} \left[\frac{(A - rA')^2}{r} + \frac{krA'^2}{1-k} \right], \quad (2.2)$$

where we have put both the gravitational constant and the speed of light equal to 1.

To obtain this expression one must transform the 3-space integral giving P^0 in (1.1) into a 2-surface integral on the boundary of this 3-space by applying Gauss theorem. To apply this theorem we need that the metric be regular enough: the 3-space derivatives of the 3-space metric must be continuous.¹ Thus, we will assume not only that, except for $r = 0$, A' in (2.1)

¹If there are no intrinsic singularities in the integration 3-volume, there always exist coordinates in which the Gauss theorem can be applied, provided that we assume, as it always done, that the differentiable manifold of the General Relativity is \mathcal{C}^2 -class by pieces. Nevertheless, in the present work, we only have to use intrinsic coordinates. Then, the \mathcal{C}^2 -class by pieces character could not be fulfilled if we are restricted to only use these intrinsic coordinates. Therefore, for intrinsic coordinates, the above regularity condition in order to apply Gauss theorem should be verified each case for the particular metric used.

exist everywhere and is different from zero, but further than A'' is continuous too, and that there is no intrinsic singularity at $r = 0$.

As remarked above, since we want this energy P^0 to be an *intrinsic* one, we must use in (2.2) an r coordinate such that dl^2 in (2.1) be asymptotically conformally flat in a manifest way.

In the next section, we calculate P^0 for two different versions of a particular LTB metric, called above the CGBH model [10, 11], which are regular enough in the above precise sense.

3 The CGBH model

In references [10, 11], the authors explore the possibility that we live close to the center of a large void, i.e. close to the center of a suitable LTB model, as an alternative to the prevailing interpretation of the Universe acceleration in terms of a Λ CDM model with a dominant dark energy component. They confront this possibility with a series of cosmological observations through two versions, the flat and the open one, of the CGBH model cited above, the first (second) version becoming asymptotically a flat (an open) FLRW without cosmological constant. The CGBH model, in its two versions, is ruled out as a result of the confrontation.

Because of this asymptotic character, comparing the LTB metric (2.1) with its flat FLRW universe limit for $r \rightarrow \infty$, having in mind the expression $ds^2 = -dt^2 + a(t)^2(dr^2 + r^2 d\sigma^2)$ for this flat FLRW metric, we easily obtain

$$A(t, r \rightarrow \infty) \sim a(t)r, \quad \lim_{r \rightarrow \infty} k(r) = 0. \quad (3.1)$$

Notice that, because of this asymptotic character, the coordinates in (2.1) are asymptotic conformally flat coordinates for dl^2 . Consequently, they are intrinsic coordinates (they fulfill the three conditions 1–3 defining the notion of intrinsic coordinates in the Introduction) and the corresponding P^0 energy we are going to calculate will be the *intrinsic* energy.

Because of (3.1), the partial contribution to P^0 , in (2.2), coming from the term containing the k function is

$$\frac{1}{2} \lim_{r \rightarrow \infty} \frac{rkA'^2}{1-k} = \frac{a^2}{2} \lim_{r \rightarrow \infty} \frac{rk}{1-k} = \frac{a^2}{2} \lim_{r \rightarrow \infty} (rk), \quad (3.2)$$

whose actual value depends on how fast k vanishes when $r \rightarrow \infty$. Then, in accordance with [10], let us define the function $\Omega_M(r)$, that generalizes the matter cosmic parameter, Ω , of the FLRW cosmology, by writing $k(r)$ like

$$k \equiv \dot{A}_0^2 (\Omega_M - 1), \quad (3.3)$$

with $\dot{A}_0 \equiv \partial_t A(t = t_0, r)$ and t_0 the cosmic present time.

The value of this function for the flat version of the CGBH model is [10]

$$\Omega_M = \Omega_{out} + (\Omega_{in} - 1) \frac{1 - \tanh[(r - r_0)/2\Delta r]}{1 + \tanh(r_0/\Delta r)}, \quad (3.4)$$

with $\Omega_{out} = 1$ ($\Omega_{out} < 1$, for the open case), where Ω_{in} , r_0 and Δr are parameters to be fitted by cosmological observations. In particular, r_0 characterizes the void size, near whose center we are assumed to be placed, and Δr the transition to uniformity.

From (3.4), we easily obtain

$$\Omega_M(r \gg \Delta r) - 1 \simeq \lambda e^{-r/\Delta r}, \quad (3.5)$$

with

$$\lambda \equiv \frac{2e^{r_0/\Delta r}}{1 + \tanh r_0/\Delta r} (\Omega_{in} - 1). \quad (3.6)$$

Then, from (3.3) and the first equation of (3.1) we have

$$k(r \gg \Delta r) \simeq \lambda \dot{a}_0^2 r^2 e^{-r/\Delta r}. \quad (3.7)$$

According to (3.2), this asymptotic behavior of k means that the contribution to P^0 in (2.2) from the term involving k vanishes. Thus, in order to calculate P^0 for the flat CGBH model, we are left with the remaining contribution from the term dealing with the function A . To calculate this contribution we have to make explicit how fast A approaches $a(t)r$ (see the first equation of (3.1)) when $r \rightarrow \infty$. The easiest way to make this is to use one of the Einstein field equations for the LTB metrics (2.1)(see [10]):

$$\dot{A}^2 + 2A\ddot{A} + k = 0, \quad (3.8)$$

$$2\ddot{A}/A + \ddot{A}'/A' = -4\pi\rho_M. \quad (3.9)$$

More precisely, we will consider Eq. (3.8) for $r \rightarrow \infty$, where as we have just seen k behaves like $k \sim r^2 e^{-r/\Delta r}$, jointly with the parametric A value for the flat CGBH model [10, 11], i.e.,

$$A(t, r) = \frac{\Omega_M}{2(1 - \Omega_M)} (\cosh \eta - 1)r, \quad (3.10)$$

$$H_0(r)t = \frac{\Omega_M}{2(1 - \Omega_M)^{3/2}} (\sinh \eta - \eta), \quad (3.11)$$

with η the parameter, and

$$H_0(r) = \frac{3}{2} H_0 (1 - \Omega_M)^{-1} [1 - \frac{\Omega_M}{(1 - \Omega_M)^{1/2}} \sinh^{-1} (\frac{1 - \Omega_M}{\Omega_M})^{1/2}], \quad (3.12)$$

with the inserted factor $3/2$ allowing us to obtain the Hubble constant H_0 , for $H_0(r \rightarrow \infty)$. Finally, Ω_M is given by (3.4) with $\Omega_{out} = 1$.

Then, in accordance to (3.1), let us write the asymptotic form of A like

$$A(t, r \rightarrow \infty) = a(t)r[1 + \epsilon(t, r)], \quad (3.13)$$

where ϵ is a function of t and r such that $\lim_{r \rightarrow \infty} \frac{\epsilon}{r^p} = 0$ for any t value and for any $p > 1$. This asymptotic expression for A would guarantee the vanishing of the corresponding contribution to the *intrinsic* value of P^0 . In the Appendix we show that A has actually such an asymptotic behavior since we obtain

$$\epsilon(t, r \rightarrow \infty) \sim e^{-r/\Delta r}. \quad (3.14)$$

Then, the resulting asymptotic form for A when $r \rightarrow \infty$ leads to

$$\lim_{r \rightarrow \infty} \frac{(A - rA')^2}{r} = 0 \quad (3.15)$$

and so to the final vanishing of the *intrinsic* energy of the flat CGBH model.

It seems, then, that the asymptotically flat CGBH cosmological model is *creatable*. However, we are going to see in the Section 5 that this vanishing of P^0 is a dramatically non stable result. We will account for this limitation by saying that the model is *physically* non *creatable*. But, before considering the instability question, in the next section we will obtain the asymptotic form of the functions $A(r, t)$ and $k(r)$ in (2.1), that is, for a general LTB metric, when the r coordinate allows us to write dl^2 asymptotically in a manifest conformal way.

4 LTB metrics in asymptotic conformally flat coordinates

Let us make a transformation of the radial coordinate r in (2.1), going to a new radial coordinate ρ , through a time independent function, that is $r \rightarrow \rho = \rho(r)$. We will choose this function such that the 3-space metric, dl^2 , becomes in the new radial coordinate asymptotically conformally flat in a manifest way. That is,

$$dl^2(\rho \rightarrow \rho_b) \simeq G^2(t, \rho) \delta_{ij} d\rho^i d\rho^j = G^2(d\rho^2 + \rho^2 d\sigma^2), \quad (4.1)$$

where ρ_i is such that $\delta_{ij} \rho^i \rho^j = \rho^2$ and where $\rho = \rho_b \equiv \text{constant}$ is the equation of the radial boundary of the space-time given by the metric ds^2 in (2.1), this equation having been assumed time independent. We can expect such a radial coordinate to exist since in [15, 16] it has been proved on general grounds for every constant time $t = t_0$. This existence is obvious in the particular case of the CGBH model since, as it is mentioned at the beginning of the Section 3, this model approaches asymptotically an open or flat FLRW universe.

Then, by simply comparing (4.1) with (2.1), we obtain

$$A'(t, \rho \rightarrow \rho_b)^2 / [1 - k(\rho \rightarrow \rho_b)] \simeq G^2(t, \rho \rightarrow \rho_b), \quad (4.2)$$

$$\rho_b^2 G^2(t, \rho \rightarrow \rho_b) \simeq A^2(t, \rho \rightarrow \rho_b). \quad (4.3)$$

We will write these two equations more compactly by putting

$$A'^2 / (1 - k) \simeq G^2, \quad A^2 \simeq \rho^2 G^2. \quad (4.4)$$

These asymptotic equations give trivially:

$$k \simeq 1 - \rho^2 A'^2 / A^2, \quad (4.5)$$

which, since k only depends on ρ , means that the function $A(t, \rho)$ factorizes asymptotically in its two arguments, that is to say, in an obvious notation:

$$A(t, \rho) \simeq a(t) f(\rho). \quad (4.6)$$

Having in mind this factorization, Eqs. (3.8) and (3.9) become asymptotically

$$\dot{a}^2 + 2a\ddot{a} = \rho^2 f'^2 / f^4 - 1 / f^2 \quad (4.7)$$

and

$$\ddot{a}/a = -\frac{4}{3}\pi\rho_M(t), \quad (4.8)$$

respectively, with $\rho_M(t) = \rho_M(t, \rho \rightarrow \rho_b)$.

From (4.7) we obtain

$$\dot{a}^2 + 2a\ddot{a} = C, \quad \rho^2 f'^2 / f^4 - 1/f^2 = C, \quad (4.9)$$

with C an arbitrary constant.

Eq. (4.8) and the first one equation of (4.9) are equivalent to the two dynamical cosmic equations for the expansion factor, $a(t)$, of a FLRW with $-C$ curvature, i.e., equivalent to (4.8) jointly with

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi\rho_M}{3} + \frac{C}{a^2}. \quad (4.10)$$

As far as the second equation of (4.9) is concerned, its general solution is a family of solutions depending on an arbitrary constant, laying aside the constant C . In all, when the parameter C vanishes we obtain the values for $a(t)$ and $f(\rho)$ corresponding to a flat FLRW universe, that is, $a(t) = \text{const.} \times t^{2/3}$, $f(\rho) = \rho$, while for $C \neq 0$ a particular solution is

$$f = \rho(1 - \frac{1}{4}C\rho^2)^{-1}, \quad (4.11)$$

giving for G the expression:

$$G^2(t, \rho) = a^2(t)(1 - \frac{1}{4}C\rho^2)^{-2}, \quad (4.12)$$

i.e., the FLRW universes with $-C$ curvature, in conformal flat 3-space coordinates:

$$ds^2 = -dt^2 + a^2(t) \frac{\delta_{ij} dx^i dx^j}{(1 - C\rho^2/4)^2}. \quad (4.13)$$

For $C > 0$ ($C < 0$) we have the open (closed) model and for $C = 0$ the flat one. Actually, from (4.13) we obtain the corresponding standard form

$$ds^2 = -dt^2 + a^2(t) \left(\frac{dr^2}{1 + Cr^2} + r^2 d\sigma^2 \right), \quad (4.14)$$

by making the coordinate transformation

$$r = \rho(1 - C\rho^2/4)^{-1}. \quad (4.15)$$

Obviously, in the particular case that $C > 0$ (the open case), we could also have obtained the result (4.14) directly from the $A(t, r)$ solution of the open CGBH model, that is from (3.10)-(3.12) and (3.4) with $\Omega_{out} < 1$.

Finally, in the open case, normalizing C to 1, we see that, while r goes from zero to infinite, ρ goes from zero to 2, so that the above boundary value ρ_b becomes $\rho_b = 2$.

5 The instability of the energy vanishing: discussion and conclusion

Let us calculate the energy P^0 of any LTB metric that behaves asymptotically like an open (non flat) FLRW metric as we have discussed in the precedent section. To make this calculation we will start from Eq. (1.1). Then, since we have assumed at the beginning of the Section 2 that our LTB metrics are regular enough, we can, using Gauss theorem, write (1.1) as the 2-surface integral on the corresponding boundary:

$$P^0 = \frac{1}{16\pi} \lim_{\rho \rightarrow \rho_b} \int (\partial_j g_{ij} - \partial_i g) n^i \rho^2 \sin \theta d\theta d\phi, \quad (5.1)$$

with $n^i \equiv \rho^i / \rho$.

Since now we have asymptotically $dl^2 \simeq G^2(t, \rho) \delta_{ij} dx^i dx^j$, P^0 becomes

$$P^0 = -\frac{1}{8\pi} \lim_{\rho \rightarrow \rho_b} \int \partial_i G^2 n^i \rho^2 \sin \theta d\theta d\phi = -\frac{1}{2} \lim_{\rho \rightarrow \rho_b} \rho^2 \partial_\rho G^2, \quad (5.2)$$

that is, according to (4.12), with C normalized to the value $C = 1$,

$$P^0 = -2 \lim_{\rho \rightarrow 2} \rho^2 \frac{d}{d\rho} (1 - \rho^2/4)^{-2} = -\infty, \quad (5.3)$$

Therefore, P^0 , whose value for the flat CGBH model (Section 3) was zero, jumps to a minus infinite value, $P^0 = -\infty$, when an elementary shift of the C constant from its original value is performed, or what is the same, when a shift from $\Omega_{out} = 1$ (see (3.4)) to the new value $\Omega_{out} < 1$ is performed as close to 1 as we want.

Thus, the vanishing of P^0 for this flat CGBH model is unstable and so this vanishing could be considered un-physical. Then, though in our parlance the flat model is strictly speaking a *creatable* one, we can consider and denote it, jointly with the open CGBH model, as physically non *creatable* universes. All this could be seen in accordance with the fact that this model is ruled out by its confrontation with cosmological observations [10, 11]. In conclusion: had this confrontation not still taken place, in view of this physical non *creatable* character, we could have predicted a subsequent negative result, not as a true prediction but as some plausible suggestion.

Furthermore, assuming a negative value for the constant C in (4.13) and normalizing to $C = -1$ we obtain the closed FLRW metric in 3-space conformal flat coordinates:

$$ds^2 = -dt^2 + a^2(t) \frac{\delta_{ij} d\rho^i d\rho^j}{(1 + \rho^2/4)^2}. \quad (5.4)$$

The boundary limit ρ_b is now $\rho_b = \infty$. Then, using the expression (5.1), we straightforwardly obtain $P^0 = 0$, that is, the LTB metric approaching asymptotically the closed FLRW model is *creatable*. Thus, we could ask if this "closed" LTB model would not be better entitled than the CGBH one to be observationally tested as a model of the void universes considered in the present paper. Actually, aside with other models, the "closed" LTB model has been tested in [9], without fully conclusive results. But, obviously, the *creatable* character of this model, by itself, could not avoid that finally the model were ruled out by observations. In fact, as stated in [9], "in practice we could always approximate the correct answer by setting Ω_k to a small nonzero value".

Finally, let us come back to Eq. (5.2). Notice that putting

$$\lim_{\rho \rightarrow \rho_b} \rho^2 \partial_\rho G^2 = 0 \quad (5.5)$$

fully characterizes the subfamily of *creatable* LTB metrics. For instance, if $\rho_b = \infty$, $G^2 \sim \frac{1}{\rho^p}$, $p > 1$, for $\rho \rightarrow \rho_b$, jointly with the particular case in which G does not depend on ρ , describe all the corresponding LTB *creatable* universes.

A Appendix. Calculating the $\epsilon(t, r)$ function

We start from (3.8), that is, $\dot{A}^2 + 2A\ddot{A} + k = 0$, and from (3.7), that is, $k(r \gg \Delta r) \simeq \lambda \dot{a}_0^2 r^2 e^{-r/\Delta r}$. Then we write A as $A = a(t)r[1 + \epsilon(t, r)]$ with $a(t) = (t/t_0)^{2/3}$. After an elementary calculation, neglecting quadratic terms in ϵ , we find

$$\frac{1}{r^2}(\dot{A}^2 + 2A\ddot{A}) \simeq (\dot{a}^2 + 2a\ddot{a})(1 + 2\epsilon) + 2a(a\ddot{\epsilon} + 3\dot{a}\dot{\epsilon}). \quad (A.1)$$

But we have $\dot{a}^2 + 2a\ddot{a} = 0$. Then, Eq. (3.8) becomes for large values of r

$$2a(a\ddot{\epsilon} + 3\dot{a}\dot{\epsilon}) \simeq -\dot{a}_0^2 \lambda e^{-r/\Delta r}, \quad (A.2)$$

that in accordance with $a(t) = (t/t_0)^{2/3}$ can be written as

$$(a^3 \dot{\epsilon})' \simeq -\frac{2}{9t_0^2} \lambda e^{-r/\Delta r}, \quad (A.3)$$

whose general solution for large values of r ($r \gg \Delta r$) is

$$\epsilon(t, r) = f(r)\alpha(t) + h(r), \quad (A.4)$$

with

$$f(r) = -\frac{2}{9t_0^2} \lambda e^{-r/\Delta r}, \quad (A.5)$$

$h(r)$ an arbitrary function, and $\alpha(t)$ the general solution of

$$(a^3 \dot{\alpha})' = a, \quad (A.6)$$

that is to say,

$$\alpha = \frac{9}{10} t_0^{4/3} t^{2/3} + \mu t_0^2 t^{-1} + \nu, \quad (A.7)$$

with μ and ν two arbitrary constants.

Substituting this expression of α in (A.4), we obtain, for large values of r but for any time,

$$\epsilon(t, r) = -\left[\frac{1}{5}a + \frac{2}{9}\left(\frac{\mu}{t} + \frac{\nu}{t_0^2}\right)\right] \lambda e^{-r/\Delta r} + h(r). \quad (A.8)$$

Then, in order to fix both arbitrary constants, μ and ν , and the arbitrary function $h(r)$, let us come back to the CGBH model, more precisely, to Eqs. (3.10) and (3.11). For small η values, that is, for small t values, we obtain

$$A(t, r) \simeq \left(\frac{3}{2}\right)^{2/3} \Omega_M^{1/3}(r) H_0^{2/3}(r) r t^{2/3}, \quad (\text{A.9})$$

where we can substitute Ω_M by its asymptotic expression from (3.5).

Furthermore, having in mind (3.4) with $\Omega_{out} = 1$ and (3.12), after an elementary calculation, we obtain the following asymptotic value

$$H(r \gg \Delta r) \simeq H_0 \left(1 - \frac{1}{5} \lambda e^{-r/\Delta r}\right). \quad (\text{A.10})$$

Carrying this expression to (A.9) we obtain for large values of r and small values of t

$$A(t, r) \simeq ar \left[1 + \frac{1}{5} (1 - a) \lambda e^{-r/\Delta r}\right]. \quad (\text{A.11})$$

Finally, through the relation $A = ar(1 + \epsilon)$, let us compare this approximated expression of $A(t, r)$ with $\epsilon(t, r)$, given by (A.8). We obtain for large values of r , but for any time,

$$\epsilon \simeq \frac{1}{5} \lambda (1 - a) e^{-r/\Delta r}, \quad (\text{A.12})$$

which is in accordance with Eq. (3.14).

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